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Question 2. (Total mark: 20) (Integrated systems modeling)

Consider the problem of dynamical modeling of a hydraulic-assisted steering wheel mechanism in a car as shown in Figure 3.

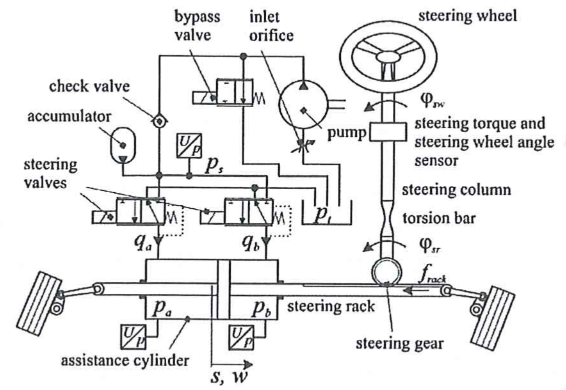

Figure 3. A hydraulic-assisted steering wheel mechanism in a car .

This steering wheel mechanism can be simplified into the multi-domain diagram in Figure 4.

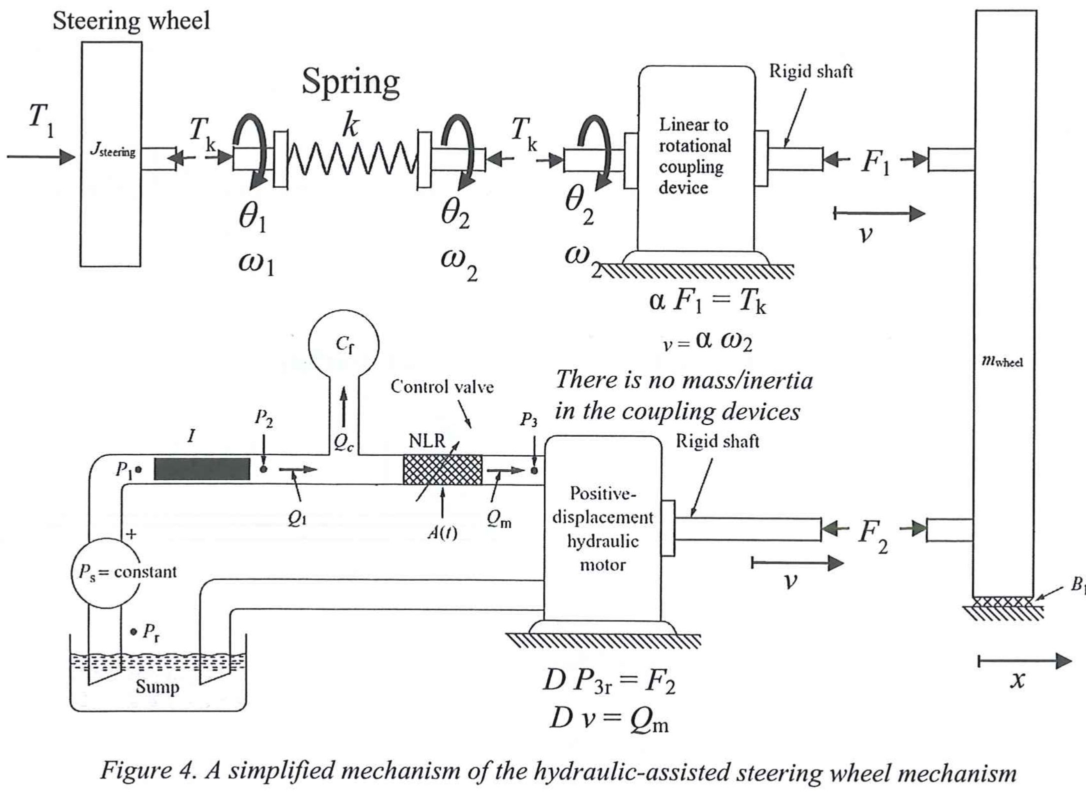

Figure 4. A simplified mechanism of the hydraulic-assisted steering wheel mechanism

In Figure 4, the driver can apply torque T_1 to the steering wheel which has inertia of $J_{inertia}$. The steering wheel will then interact with the steering rack (see also Figure 3) through a torsion spring and a linear-to-rotational coupling device. The spring has a spring constant of k, while the linear-to-rotational coupling device transforms the spring torque T_k and angular velocity ω_2 into linear force F_1 and linear velocity v by the following formula:

$$\alpha\,\omega_2 = v \quad \text{and} \quad T_k = \alpha\,F_1$$

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On the other hand, the pressure source P_s is used to provide the "hydraulic-assisted" forces to the wheel which is regulated through the control valve NLR.

The pressure across the valve NLR and the fluid flow through the valve NLR is given by

$$Q_m = cA\sqrt{P_{23}}$$

or equivalently,

$$\frac{1}{(cA)^2}Q_m{}^2 = P_{23}$$

where c is a valve constant and A is the area of opening of the valve. For providing the required assisted torque, the opening of control valve NLR, which is denoted by A, is regulated based on the amount of spring torsion in the steering wheel bar T_2, i.e., $A = f(T_2)$ for some function f.

The fluid inertor has an inertance of I and the fluid capacitor has the capacitance of C_f. The pressure across the coupling device P_{3r} and the fluid flow through the coupling device Q_m is converted to a linear force F_2 and a linear velocity v by the following formula

$$D\,P_{3r} = F_2 \text{ and } D\,v = Q_m$$

The linear force due to the driver torque F_1 and due to the hydraulic pressure F_2 drive the wheel mass m_{wheel} which interacts with the road through friction. The friction is modeled by a damper with the damping coefficient of B_1.

Write the state-space equations of the complete systems with the input $\begin{bmatrix} T_1 \\ P_s \end{bmatrix}$, the state $\begin{bmatrix} Q_1 \\ P_{2r} \\ F_k \\ \omega_1 \\ v \end{bmatrix}$ and the measured output of torsion torque T_k! **(20 marks)**

$$I\frac{dQ_1}{dt} = P_{12} = P_s - P_{23} - P_{3r}$$

$$= P_s - \frac{1}{(cA)^2}Q_m^2 - \frac{1}{D}F_2$$

$$= P_s - \frac{1}{(cA)^2}D^2v^2 - \frac{1}{D}F_2 \qquad (1)$$

$$C_f\frac{dP_{3r}}{dt} = Q_c = Q_1 - Q_m = Q_1 - Dv \qquad (2)$$

$$m_{wheel}\frac{dv}{dt} = F_2 + F_1 - B_1 v \qquad (3)$$

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$$J_{steering} \frac{d\omega_1}{dt} = T_1 - T_k \quad (4)$$

$$\frac{1}{k} \frac{dT_k}{dt} = \omega_1 - \omega_2 = \omega_1 - \frac{1}{\alpha} v \quad (5)$$

From (3), we get

$$m_{wheel} \frac{dv}{dt} = F_2 + \frac{1}{\alpha} T_k - B_1 v$$

$$F_2 = m_{wheel} \frac{dv}{dt} - \frac{1}{\alpha} T_k + B_1 v$$

Substituting this to (1), we arrive at

$$I \frac{dQ_1}{dt} = P_s - \frac{1}{(CA)^2} D^2 v^2 - \frac{1}{D}\left(m_{wheel} \frac{dv}{dt} - \frac{1}{\alpha} T_k + B_1 v\right)$$

$$I \frac{dQ_1}{dt} + \frac{m_{wheel}}{D} \frac{dv}{dt} = -\frac{1}{(CA)^2} D^2 v^2 + \frac{1}{\alpha D} T_k - \frac{B_1}{D} v + P_s$$

$$\begin{bmatrix} I & 0 & 0 & 0 & \frac{m_{wheel}}{D} \\ 0 & C_f & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{k} & 0 & 0 \\ 0 & 0 & 0 & J_{steering} & 0 \end{bmatrix} \begin{bmatrix} \frac{dQ_1}{dt} \\ \frac{dP_{2r}}{dt} \\ \frac{dT_k}{dt} \\ \frac{d\omega_1}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{(CA)^2} D^2 v^2 + \frac{1}{\alpha D} T_k - \frac{B_1}{D} v + P_s \\ Q_1 - Dv \\ \omega_1 - \frac{1}{\alpha} v \\ T_1 - T_k \end{bmatrix}$$

Or alternatively, we modify (1) into

$$I \frac{dQ_1}{dt} = P_s - P_{2r} \quad (1)$$

and modify (3) into

$$m_{wheel} \frac{dv}{dt} = D P_{3r} + \frac{1}{\alpha} T_k - B_1 v$$

$$= D(-P_{23} + P_{2r}) + \frac{1}{\alpha} T_k - B_1 v \quad (3)$$

$$= D\left(-\frac{1}{(CA)^2} D^2 v^2 + P_{2r}\right) + \frac{1}{\alpha} T_k - B_1 v$$

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Question 3. (Total mark: 20) (State-space modeling, linearization, delay)

Consider a simple model of hydraulic drilling machine as shown in Figure 5. The constant flow source Q_s which is controlled by the control valve NLR is used to drive the hydraulic motor (via the coupling device) for rotating the drilling bit. There is a fluid capacitor with capacitance of C_f.

The pressure across the valve NLR and the fluid flow through the valve NLR is given by

$$\frac{1}{(cA)^2}{Q_m}^2 = P_{12}$$

where c is a valve constant and A is the area of opening of the valve which will be the control input of the drilling machine.

The pressure across the coupling device P_{2r} and the fluid flow through the coupling device Q_m is converted to a linear force F_2 and a linear velocity v by the following formula

$$D\,P_{2r} = T \text{ and } D\,\omega = Q_m$$

The generated torque rotates the drilling bit with an inertia of J_{drill} and it interacts with the soil through friction force, which has a damping constant of B_1.

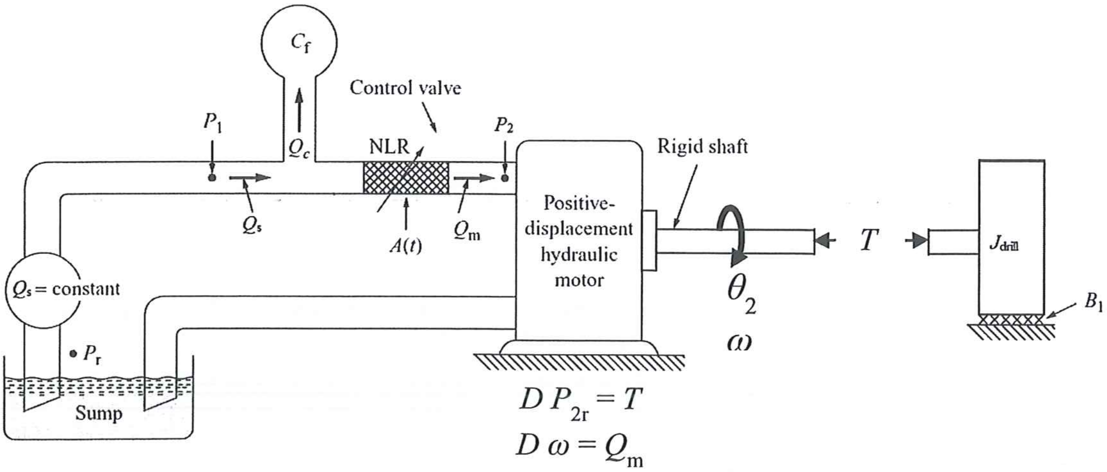

Figure 5. A simple electrical-mechanical system.

a) If the input u is given by area A, the flow source Q_s is assumed to be a constant 2 (i.e., $Q_s = 2$) and the measured output y is the angular velocity ω, show that the state space equation of the system can be given by

$$\begin{bmatrix} \dot{P}_{1r} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \frac{1}{C_f}(-D\omega + 2) \\ \frac{1}{J_{drill}}\left(DP_{1r} - \frac{D^3}{(cu)^2}\omega^2 - B_1\omega\right) \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} P_{1r} \\ \omega \end{bmatrix}$$

(5 marks)

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$$C_f \frac{dP_{1r}}{dt} = Q_s - Q_m = Q_s - D\omega$$

$-B_1\omega$ $\quad$ $-B_1\omega$

$$J_{drill} \frac{d\omega}{dt} = T = DP_{2r} = D(P_{1r} - P_{12}) - B_1\omega$$

$$= D\left(P_{1r} - \frac{1}{(cA)^2} Q_m^2\right) - B_1\omega$$

$$= DP_{1r} - \frac{1}{(cA)^2} D^3\omega^2 - B_1\omega$$

Putting these to a combined state equation, we have

$$\dot{P}_{1r} = \frac{1}{C_f} Q_s - \frac{D}{C_f}\omega = \frac{2}{C_f} - \frac{D}{C_f}\omega$$

$$\dot{\omega} = \frac{D}{J_{drill}} P_{1r} - \frac{D^3}{(cA)^2 J_{drill}} \omega^2 - B_1\omega$$

and the measured output is

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} P_{1r} \\ \omega \end{bmatrix}$$

b) Suppose now that the following numerical values are used for the state-space as above: $C_f = 1$, $J_{drill} = 1$, $c = 1$, $B_1 = 2$, $D = 2$;
Show that the linearization of the system around the operating point ($\omega^* = 1$, $P_{1r}^* = 2, u^* = 2$) is given by

$$\begin{bmatrix} \dot{\tilde{P}}_{1r} \\ \dot{\tilde{\omega}} \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} \tilde{P}_{1r} \\ \tilde{\omega} \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \tilde{u}$$

$$e = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{P}_{1r} \\ \tilde{\omega} \end{bmatrix}$$

where $\tilde{P}_{1r} = P_{1r} - P_{1r}^*, \tilde{\omega} = \omega - \omega^*, \tilde{u} = u - u^*$ and $e = \tilde{\omega}$. **(5 marks)**

Using these numerical values, we have (from (a))

$$\dot{P}_{1r} = -2\omega + 2$$

$$\dot{\omega} = 2P_{1r} - \frac{8}{u^2}\omega^2 - 2\omega$$

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Linearizing it around $\omega^* = 1$, $P_{ir}^* = 2$ & $u^* = 2$, we get

and using $\tilde{P}_{ir} = P_{ir} - P_{ir}^*$
$\tilde{\omega} = \omega - \omega^*$
$\tilde{u} = u - u^*$

$$\dot{\tilde{P}}_{ir} = -2\tilde{\omega}$$

$$\dot{\tilde{\omega}} = 2\tilde{P}_{ir} - \frac{\partial}{\partial \omega}\left(\frac{8}{u^2}\omega^2\right)\Bigg|_{\substack{\omega^* \\ u^*}}\tilde{\omega} - \frac{\partial}{\partial u}\left(\frac{8}{u^2}\omega^2\right)\Bigg|_{\substack{\omega^* \\ u^*}}\tilde{u} - 2\tilde{\omega}$$

$$= 2\tilde{P}_{ir} - \frac{16\omega^*}{(u^*)^2}\tilde{\omega} + \frac{16}{(u^*)^3}(\omega^*)^2\tilde{u} - 2\tilde{\omega}$$

$$= 2\tilde{P}_{ir} - 4\tilde{\omega} + 2\tilde{u} - 2\tilde{\omega}$$

$$= 2\tilde{P}_{ir} - 6\tilde{\omega} + 2\tilde{u}$$

Putting everything into state-space matrix form:

$$\begin{bmatrix} \dot{\tilde{P}}_{ir} \\ \dot{\tilde{\omega}} \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & -6 \end{bmatrix}\begin{bmatrix} \tilde{P}_{ir} \\ \tilde{\omega} \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\tilde{u}$$

$$e = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{P}_{ir} \\ \tilde{\omega} \end{bmatrix}$$

c) Based on the answer in a), show that the transfer function from $\tilde{u}$ to e is given by

$$\frac{E(s)}{\tilde{U}(s)} = \frac{2s}{s^2 + 6s + 4}$$

(5 marks)

$$\frac{E(s)}{\tilde{U}(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s & 2 \\ -2 & s+6 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{s^2 + 6s + 4}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s+6 & -2 \\ 2 & s \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \frac{2s}{s^2 + 6s + 4}$$

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d) If a proportional+integral feedback control is used in the linearized equation where $C(s) = 2 + \frac{1}{s}$, estimate the critical time delay that is tolerable for the stability of the closed-loop system! **(5 marks)**

Using $G(s) = \frac{E(s)}{Q(s)}$

$$C(s)G(s) = \frac{2s}{s^2+6s+4} \cdot \frac{2s+1}{s} = \frac{4s+2}{s^2+6s+4}$$

For getting critical time delay, we need to compute the phase margin as follows.

The freq. response is given by

$$C(\omega i)\, G(\omega i) = \frac{4\omega i + 2}{-\omega^2 + 4 + 6\omega i}$$

The amplitude is given by

$$|C(\omega i)\, G(\omega i)| = \frac{\sqrt{4 + 16\omega^2}}{\sqrt{(4-\omega^2)^2 + 36\omega^2}}$$

The frequency ω^* where the amplitude is one is given by

$$\sqrt{4 + 16(\omega^*)^2} = \sqrt{(4-\omega^{*2})^2 + 36(\omega^*)^2}$$

$\Rightarrow \quad 4 + 16\omega^{*2} = 16 - 8\omega^{*2} + \omega^{*4} + 36\omega^{*2}$

$\Rightarrow \quad \cancel{21\omega^{*2} + 8\omega^{*} + 12 = 0} \qquad \omega^{*4} + 12\omega^{*2} + 12 = 0$

$\cancel{\omega^* = \frac{8}{42} \pm \frac{\sqrt{64 - 24 \times 21}}{42}}$

~~which are complex numbers~~

$$\omega^{*2} = \frac{-12}{2} \pm \frac{1}{2}\sqrt{144 - 48}$$

$$= -6 \pm \frac{\sqrt{96}}{2} \approx -6 \pm 4.9$$

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ω^* is complex number which implies that there is no freq ω^* where the gain is equal to one.

Then the system has an infinite phase margin (P_m)

Since the critical time delay is calculated from

$$-T_{critical}\,\omega^* + \angle G(\omega^* i)C(\omega^* i) = -\pi$$

and since ω^* is not a real number, there is no critical-time delay in the closed-loop systems.

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Question 5. (Total mark: 20) (State-feedback and state-observer)

Consider again the system as given in the Question 3a) where the linearized system is described as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Design an optimal state feedback which minimizes the following cost function:

$$J = \int_0^\infty 3x_1^2(\tau) + 2x_1(\tau)x_2(\tau) + 14x_2^2(\tau) + u^2(\tau)d\tau .$$ **(10 marks)**

Let us compute the solution to the LQR problem. Denote $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$ and from the cost function, it can be deduced that $Q = \begin{bmatrix} 3 & 1 \\ 1 & 14 \end{bmatrix}$ and $R = 1$.

Then from the formula $A^TP + PA - PBR^{-1}B^TP + Q = 0$ we have

$$\begin{bmatrix} 0 & 2 \\ -2 & -6 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 2 & -6 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & 14 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 2p_2 & 2p_3 \\ -2p_1-6p_2 & -2p_2-6p_3 \end{bmatrix} + \begin{bmatrix} 2p_2 & -2p_1-6p_2 \\ 2p_3 & -2p_2-6p_3 \end{bmatrix} - \begin{bmatrix} 4p_2^2 & 4p_2p_3 \\ 4p_2p_3 & 4p_3^2 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 1 & 14 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 4p_2 - 4p_2^2 + 3 & 2p_3 - 2p_1 - 6p_2 - 4p_2p_3 + 1 \\ 2p_3 - 2p_1 - 6p_2 - 4p_2p_3 + 1 & -4p_2 - 12p_3 - 4p_3^2 + 14 \end{bmatrix} = 0$$

There are three equations to be solved, namely:

$$4p_2 - 4p_2^2 + 3 = 0 \qquad (1)$$

$$-4p_2 - 12p_3 - 4p_3^2 + 14 = 0 \qquad (2)$$

and

$$2p_3 - 2p_1 - 6p_2 - 4p_2p_3 + 1 = 0 \qquad (3)$$

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Solving (1) we get

$$-4p_2^2 + 4p_2 + 3 = -(2p_2 + 1)(2p_2 - 3) = 0$$

Hence $p_2 = -\frac{1}{2}$ or $p_2 = \frac{3}{2}$

Take $p_2 = -\frac{1}{2}$ (since the other root does not lead to a positive definite P if we carry on using $p_2 = \frac{3}{2}$).

then, we can solve (2) as follows:

$$4p_3^2 + 12p_3 - 14 - 2 = 4p_3^2 + 12p_3 - 16 = 0$$

$$\Rightarrow (2p_3 - 2)(2p_3 + 8) = 0$$

which means we have two roots $p_3 = 1$ and $p_3 = -4$.

Since we want to have a positive definite P, the principal diagonal must be positive, hence $p_3 = 1$ is chosen (the principal diagonal of P are p_1 & p_3). Using $p_3 = 1$ & $p_2 = -\frac{1}{2}$, we can solve (3) as follows

$$2 - 2p_1 + 3 + 2 + 1 = 0$$

$$p_1 = \frac{8}{2} = 4.$$

Hence $P = \begin{bmatrix} 4 & -1/2 \\ -\frac{1}{2} & 1 \end{bmatrix}$ which is positive definite.

The control law is then given by

$$u = -R^{-1}B^TPx = -\begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} 4 & -1/2 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1 - 2x_2.$$

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b) Let a state observer be designed for the above system where it has the following form:

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{bmatrix} = \begin{bmatrix}0 & -2\\ 2 & -6\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}0\\ 2\end{bmatrix}u + L(y-\hat{y})$$

$$\hat{y} = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}$$

where $\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}$ is the estimated state, $\hat{y}$ is the estimated output and L is the observer gain. Design the observer gain L, such that the dynamics of the estimation error has eigenvalues at -10 and -15, i.e., the desired characteristics polynomial of the error dynamics is given by

$$\lambda^2 + 25\lambda + 150 = 0$$

(10 marks)

The error estimation dynamics is given by

$$\begin{bmatrix}\dot{\tilde{x}}_1\\ \dot{\tilde{x}}_2\end{bmatrix} = \begin{bmatrix}0 & -2\\ 2 & -6\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix} - \begin{bmatrix}l_1\\ l_2\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix}$$

where $\tilde{x}_1 = x_1 - \hat{x}_1$ and $\tilde{x}_2 = x_2 - \hat{x}_2$.

In a compact form, we have it as follows

$$\begin{bmatrix}\dot{\tilde{x}}_1\\ \dot{\tilde{x}}_2\end{bmatrix} = \begin{bmatrix}0 & -2-l_1\\ 2 & -6-l_2\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix}$$

The characteristic polynomial is given by

$$\left|\left[\lambda I - \begin{bmatrix}0 & -2-l_1\\ 2 & -6-l_2\end{bmatrix}\right]\right| = \left|\begin{bmatrix}\lambda & 2+l_1\\ -2 & \lambda+6+l_2\end{bmatrix}\right|$$

$$= \lambda^2 + (6+l_2)\lambda + 4 + 2l_1$$

Equating the desired characteristic polynomial in the question with the above one, we arrive at

$$6+l_2 = 25 \Rightarrow l_2 = 19$$

and $$4+2l_1 = 150 \Rightarrow l_1 = 73$$

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Question 6. (Total mark: 20) (Mechanical systems modeling)

Consider the problem of dynamical modeling of a flexible conveyor belt as shown in Figure 7.

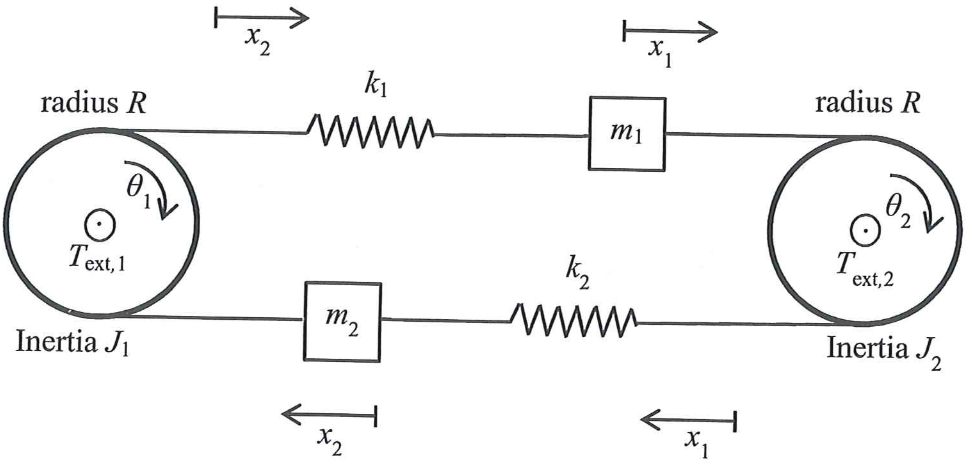
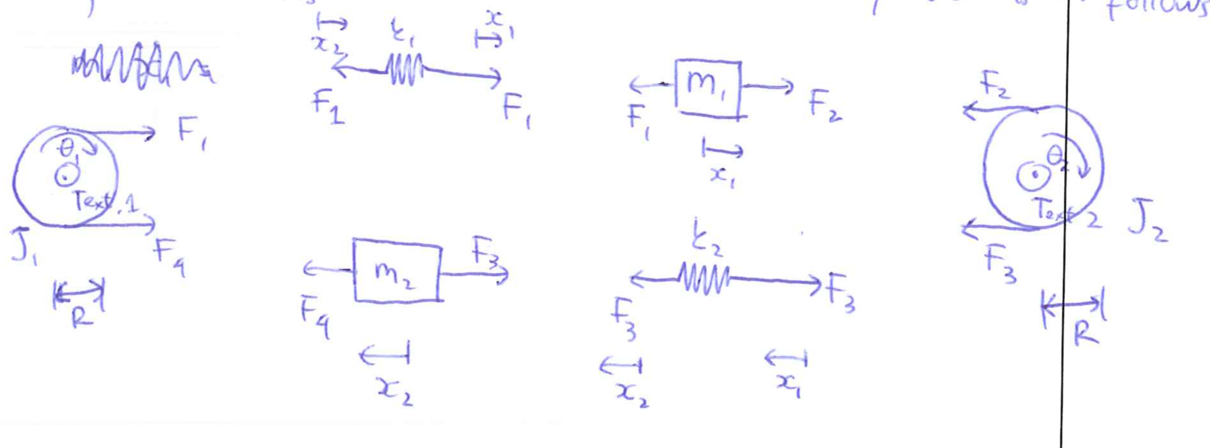


Figure 7. Modeling of a flexible conveyor belt with two masses, m_1 and m_2.

In the conveyor belt system as in Figure 7, two masses m_1 and m_2, are driven by two pulleys where each of these are driven by a motor with torque of $T_{ext,1}$ and $T_{ext,2}$, respectively. The conveyor is considered to be flexible and is modeled by springs with spring constants, k_1 and k_2, respectively.

With a reference to the illustration in Figure 7, derive the dynamical modeling of the system (either via classical Newton's laws or via Euler-Lagrange formalism) with the generalized coordinate of (θ_1, θ_2) and the generalized forces are given by $T_{ext,1}$ and $T_{ext,2}$.

Using Newtons : Let us consider every elements as follows



Then for every components we have

$1\frac{1}{2}$ ① $m_1 \ddot{x}_1 = F_2 - F_1$ $1\frac{1}{2}$ ③ $F_1 = k_1 (x_1 - x_2)$ $1\frac{1}{2}$ ⑤ $J_1 \ddot{\theta}_1 = F_1 R - F_4 R$

$1\frac{1}{2}$ ② $m_2 \ddot{x}_2 = F_4 - F_3$ $1\frac{1}{2}$ ④ $F_3 = k_2 (x_2 - x_1)$ $1\frac{1}{2}$ ⑥ $J_2 \ddot{\theta}_2 = F_3 R - F_2 R$

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Substituting F_1 & F_3 to every masses & inertias we get

(1) (1) $m_1 \ddot{x}_1 = F_2 - k_1(x_1 - x_2)$

(1) (1) $m_2 \ddot{x}_2 = F_4 - k_2(x_2 - x_1)$

(1) $J_1 \ddot{\theta}_1 = k_1 R(x_1 - x_2) - F_4 R + T_{ext,1}$

(1) $J_2 \ddot{\theta}_2 = k_2 R(x_2 - x_1) - F_2 R + T_{ext,2}$

(1) Noticing that $x_1 = \theta_2 R$ and $x_2 = \theta_1 R$ by the standard formula for radians angle, and by substituting F_2 & F_4, we get

(3)
$$J_1 \ddot{\theta}_1 = k_1 R(\theta_2 R - \theta_1 R) - R\left(m_2 R \ddot{\theta}_1 + k_2(R\theta_1 - R\theta_2)\right) + T_{ext,1}$$

$$= k_1 R^2 \theta_2 - k_1 R^2 \theta_1 - m_2 R^2 \ddot{\theta}_1 - k_2 R^2 \theta_1 + R^2 k_2 \theta_2 + T_{ext,1}$$

$$\Rightarrow J_1 \ddot{\theta}_1 + m_2 R^2 \ddot{\theta}_1 + (k_1 R^2 + k_2 R^2)\theta_1 - (k_1 R^2 + k_2 R^2)\theta_2 = T_{ext,1}$$

(3)
$$J_2 \ddot{\theta}_2 = k_2 R(\theta_1 R - \theta_2 R) - R\left(m_1 R \ddot{\theta}_2 + k_1(R\theta_2 - R\theta_1)\right) + T_{ext,2}$$

$$= k_2 R^2 \theta_1 - k_2 R^2 \theta_2 - m_1 R^2 \ddot{\theta}_2 - k_1 R^2 \theta_2 + k_1 R^2 \theta_1 + T_{ext,2}$$

$$\Rightarrow J_2 \ddot{\theta}_2 + m_1 R^2 \ddot{\theta}_2 + (k_2 R^2 + k_1 R^2)\theta_2 - (k_2 R^2 + k_1 R^2)\theta_1 = T_{ext,2}$$

Using the Euler-Lagrange formalism:

Notice that
$$x_1 = \theta_2 R \qquad x_2 = \theta_1 R$$
$$\dot{x}_1 = \dot{\theta}_2 R \qquad \dot{x}_2 = \dot{\theta}_1 R$$
$$\ddot{x}_1 = \ddot{\theta}_2 R \qquad \ddot{x}_2 = \ddot{\theta}_1 R$$

The kinetic energy (in the generalized coordinate θ_1, θ_2) is

$$E_k = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2$$ (4)

$$= \frac{1}{2} m_1 R^2 \dot{\theta}_2^2 + \frac{1}{2} m_2 R^2 \dot{\theta}_1^2 + \frac{1}{2} J_1 \dot{\theta}_1^2 + \frac{1}{2} J_2 \dot{\theta}_2^2$$

$$= \frac{1}{2}(m_2 R^2 + J_1)\dot{\theta}_1^2 + \frac{1}{2}(m_1 R^2 + J_2)\dot{\theta}_2^2$$

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The potential energy from the spring:

$$E_p = \frac{1}{2} k_1 (x_1 - x_2)^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

$$= \frac{1}{2} k_1 R^2 (\theta_2 - \theta_1)^2 + \frac{1}{2} k_2 R^2 (\theta_1 - \theta_2)^2 \qquad 4$$

The Lagrangian is then given by

$$L = E_k - E_p = \frac{1}{2}(m_2 R^2 + J_1)\dot{\theta}_1^2 + \frac{1}{2}(m_1 R^2 + J_2)\dot{\theta}_2^2$$

$$- \frac{1}{2} k_1 R^2 (\theta_2 - \theta_1)^2 - \frac{1}{2} k_2 R^2 (\theta_1 - \theta_2)^2 \qquad 1$$

The Euler-Lagrange eqs:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = T_{ext,1} \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = T_{ext,2} \qquad (2) \qquad 1$$

For calculating (1) we have the following:

$$\frac{d}{dt}\left((m_2 R^2 + J_1)\dot{\theta}_1\right) - \left(-k_1 R^2 (\theta_2 - \theta_1)(-1) - k_2 R^2 (\theta_1 - \theta_2)\right) = T_{ext,1}$$

$$\Rightarrow \quad (m_2 R^2 + J_1)\ddot{\theta}_1 + (k_1 R^2 + k_2 R^2)\theta_1 - (k_1 R^2 + k_2 R^2)\theta_2 = T_{ext,1} \qquad 5$$

Similarly, for (2), we have the following:

$$\frac{d}{dt}\left((m_1 R^2 + J_2)\dot{\theta}_2\right) - \left(-k_1 R^2 (\theta_2 - \theta_1) - k_2 R^2 (\theta_1 - \theta_2)(-1)\right) = T_{ext,2}$$

$$5$$

$$\Rightarrow \quad (m_1 R^2 + J_2)\ddot{\theta}_2 + (k_1 R^2 + k_2 R^2)\theta_2 - (k_1 R^2 + k_2 R^2)\theta_1 = T_{ext,2}$$

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Question 7. (Total mark: 20) (Time-discretization and z-transform)

a) Consider again the drilling system where the transfer function of its linearization is given in Question 3c, i.e.,

$$\frac{E(s)}{\tilde{U}(s)} = \frac{s}{s^2+6s+4}$$

Using the bilinear transformation, calculate the discrete-time transfer function (with sampling period of T = 0.2sec) and then compute the corresponding difference equation (associated with the obtained discrete-time transfer function)! **(5 marks)**

$$\frac{E(z)}{\tilde{U}(z)} = \frac{\frac{2}{0.2}\frac{1-z^{-1}}{1+z^{-1}}}{\left(\frac{2}{0.2}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 6\left(\frac{2}{0.2}\frac{1-z^{-1}}{1+z^{-1}}\right)+4} = \frac{10\frac{1-z^{-1}}{1+z^{-1}}}{100\frac{1-2z^{-1}+z^{-2}}{1+2z^{-1}+z^{-2}} + 60\frac{1-z^{-1}}{1+z^{-1}}+4}$$

$$= \frac{10(1-z^{-2})}{100(1-2z^{-1}+z^{-2})+60(1-z^{-2})+4}$$

$$= \frac{10-10z^{-2}}{40z^{-2}-200z^{-1}+164}$$

The corresponding difference equation is obtained via inverse z-transform to the following equation:

$$(40z^{-2}-200z^{-1}+164)E(z) = (10-10z^{-2})\tilde{U}(z)$$

$$\mathcal{Z}^{-1} \Rightarrow 164\,e(k) - 200\,e(k-1) + 40\,e(k-2) = 10\tilde{u}(k) - 10\tilde{u}(k-2)$$

b) Calculate the discrete-time transfer function (via z-transform) associated with the following difference equation

$$y(k) - 1.5y(k-1) - y(k-2) = u(k) + 0.5u(k-1)\,!$$

(5 marks)

$$Y(z) - 1.5z^{-1}Y(z) - z^{-2}Y(z) = U(z) + 0.5z^{-1}U(z)$$

$$\frac{Y(z)}{U(z)} = \frac{1+0.5z^{-1}}{1-1.5z^{-1}-z^{-2}}$$

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c) Explain whether the discrete-time system in Question 7b) is stable or not!

(5 marks)

The denominator polynomial (which is the characteristic polynomial in z-domain) is given by $X(z) = 1 - 1.5z^{-1} - z^{-2} = (2 + z^{-1})(\frac{1}{2} - z^{-1})$

The roots of this equation are then

$$z = -\frac{1}{2} \quad \text{and} \quad z = 2 .$$

Since one of the roots is outside the unit circle, then the system is unstable.

Remember !!! that a discrete-time system is stable if the pole is located inside a unit circle.

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d) Suppose that we want to identify the parameters of a second-order mechanical system given below

$$m\ddot{x} + b\dot{x} + kx = F$$

where m is the mass, b is the damping constant, k is the spring constant, x is the displacement and F is the force.
In order to do that, we can use the discrete-time approximation of the system and identify the parameters based on it. Using the Euler-approximation, compute the discrete-time approximation of the above dynamical system!

(5 marks)

~~Obtain~~ Note that by applying Laplace, we get :

$$(ms^2 + bs + k)X(s) = F(s)$$

Using the Euler approximation for transforming from s-domain to the z-domain, we get

$$\left(m\left(\frac{1}{T}(1-z^{-1})\right)^2 + b\left(\frac{1}{T}(1-z^{-1})\right) + k\right)X(z) = F(z)$$

$$\Rightarrow \left(\frac{m}{T^2}(1-2z^{-1}+z^{-2}) + \frac{b}{T}(1-z^{-1}) + k\right)X(z) = F(z)$$

$$\Rightarrow \left(\frac{m}{T^2}z^{-2} - \left(\frac{2m}{T^2} + \frac{b}{T}\right)z^{-1} + \frac{m}{T^2} + \frac{b}{T} + k\right)X(z) = F(z)$$

Using the inverse z-transform, we get.

$$\left(\frac{m}{T^2} + \frac{b}{T} + k\right)x(k) - \left(\frac{2m}{T^2} + \frac{b}{T}\right)x(k-1) + \frac{m}{T^2}x(k-2) = F(k).$$